ME 4555 - Lecture 29 - Bode plots

A Bode plot (pronounced "Boh-dee") is a pair of plots ($M(\omega)$ and $\phi(\omega)$) displayed in a particular format.

Magnitude is plotted in units of decibels (dB) on a log scale for ω.
Phase is plotted in degrees on a log scale for ω.
The frequency ω can be measured in rad/sec, Hz, rpm, etc. rad/sec is typical.

named after Alexander Graham Bell, inventor of the telephone!

What is a decibel? it's $\frac{1}{10}$ of a bel, which is $\log_{10}\left(\frac{\text{Power out}}{\text{Power in}}\right)$. Historically used to measure sound degradation/amplification. The human ear is sensitive to multiplicative increases, i.e. going from $P=1$ to $P=10$ sounds the same as going from $P=10$ to $P=100$. In both cases, it is 10x, or 1 B (bel), or 10 dB ← (decibel). Power is proportional to the square of magnitude. Therefore, we have:

$$
\begin{aligned}
(\text{gain in dB}) &= 10\cdot(\text{gain in B}) \\
&= 10\cdot\log_{10}\left(\frac{P_{out}}{P_{in}}\right) \\
&= 10\cdot\log_{10}\left(\frac{M_{out}^2}{M_{in}^2}\right) \\
&= 20\log_{10}\left(\frac{M_{out}}{M_{in}}\right)
\end{aligned}
$$

remember: $\log(xy) = \log x + \log y$ and $\log(x^n) = n\log x$

So $\underbrace{M_{dB}}_{\text{Magnitude in dB}} = 20\log_{10} M = 20\log_{10}|G(j\omega)|$

Commonly used amplifications

M	M_{dB}
100	40 dB
10	20 dB
1	0 dB
0.1	−20 dB
0.01	−40 dB
2	$\approx$ 6 dB
$1.414 \approx \sqrt{2}$	$\approx$ 3 dB
$0.707 \approx \frac{1}{\sqrt{2}}$	$\approx$ −3 dB
$\frac{1}{2}$	$\approx$ −6 dB.

Remember dB is a log scale!

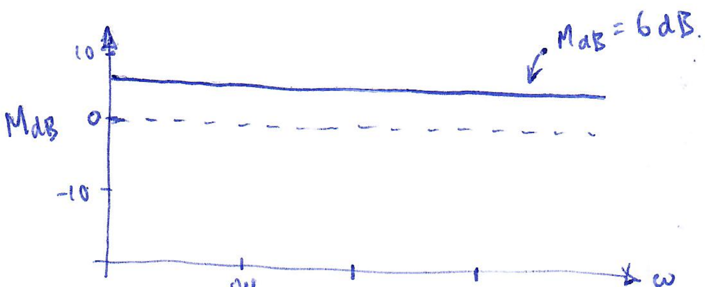
- ★ so multiplication of gains results in addition on the Bode plot.
- ★ inverting a gain, e.g. $\frac{1}{2}$ instead of 2 flips the sign in dB, so −6 dB instead of 6 dB
- ★ a gain of 0 corresponds to $-\infty$ dB.

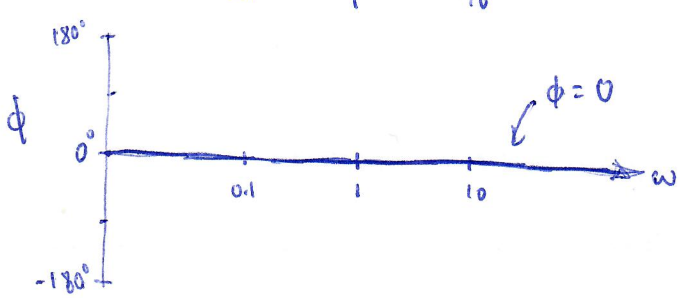
Ex. Bode plot of a constant gain

Consider $G(j\omega) = 2$ (gain of 2)

$M(\omega) = 2$. so $M_{dB} = 6$ dB.

$\phi(\omega) = 0°$.



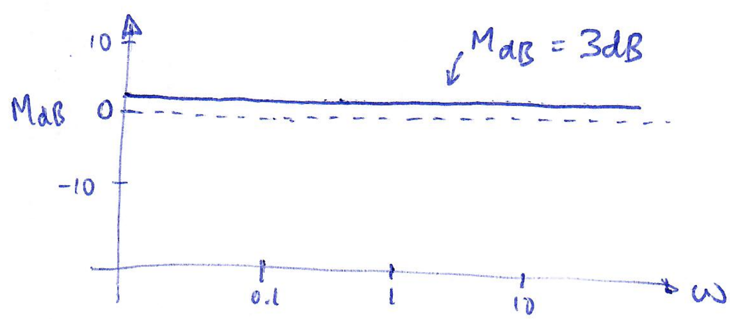


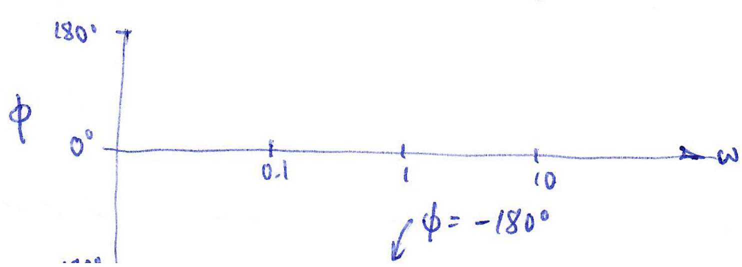
what about a negative gain?

Consider $G(j\omega) = -1.414$. Think of this as a complex number. Magnitude always ≥ 0.

$M(\omega) = 1.414$, so $M_{dB} = 3$ dB.

$\phi(\omega) = -\pi = -180°$ (+180° also correct)





Ex: 1st order system, $G(s) = \frac{K}{\tau s + 1}$. What is the Bode plot?

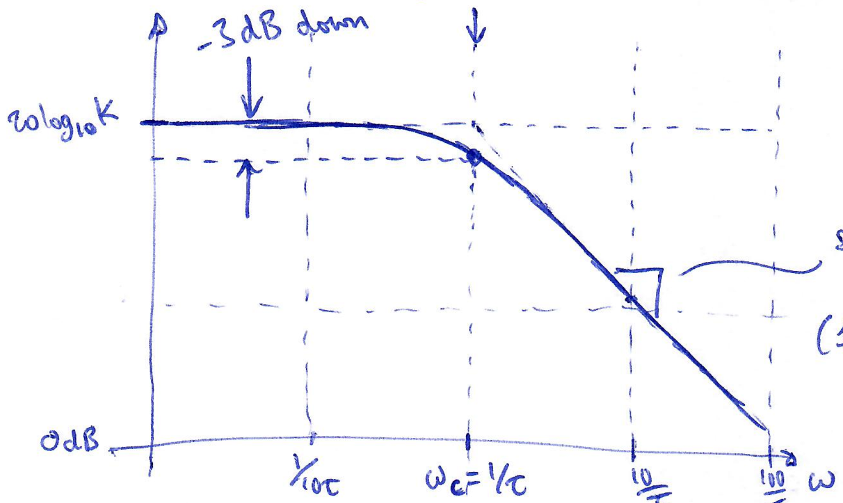
Let's start by calculating the magnitude and phase of the frequency response:

$$G(j\omega) = \frac{K}{j\omega\tau + 1} = \frac{K(1 - j\omega\tau)}{(1 + j\omega\tau)(1 - j\omega\tau)} = \frac{K(1 - j\omega\tau)}{1 + \omega^2\tau^2} = \underbrace{\left(\frac{K}{\tau^2\omega^2 + 1}\right)}_{p(\omega)} + j\underbrace{\left(\frac{-K\tau\omega}{\tau^2\omega^2 + 1}\right)}_{q(\omega)}$$

Therefore,
$$\begin{cases} M(\omega) = \sqrt{p(\omega)^2 + q(\omega)^2} = \dfrac{K}{\sqrt{(\tau\omega)^2 + 1}} \\ \phi(\omega) = -\arctan(\omega\tau) \end{cases}$$

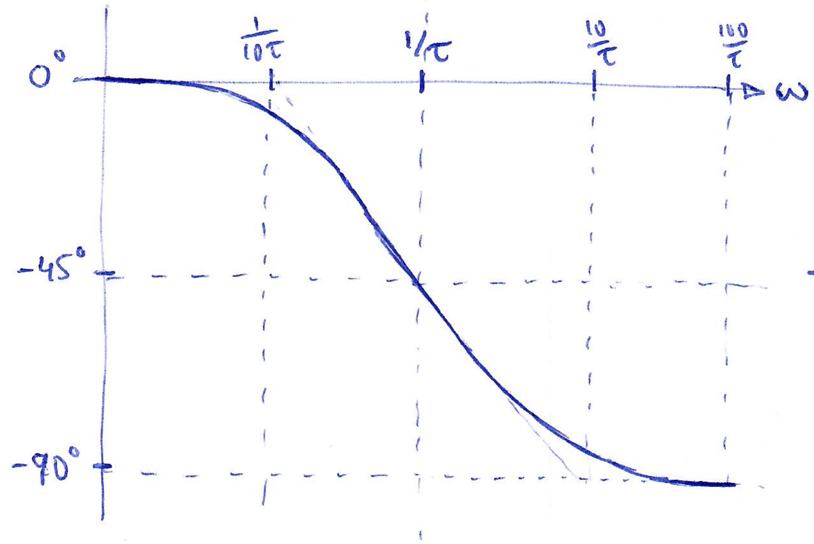
Bode plot:

$\omega_c = 1/\tau$ = "corner frequency".



slope = -20 dB per decade

(1 decade = 10x frequency)



☆ At $\omega \ll \omega_c$, $\phi \approx 0°$

$M \approx K$, so $M_{dB} \approx 20\log_{10}K$.

☆ At $\omega_c = 1/\tau$, $\phi = -45°$

and $M = \frac{K}{\sqrt{2}}$. therefore,

$$M_{dB} \approx 20\log_{10}K \underbrace{- 3}_{\text{3dB drop} \approx \frac{1}{\sqrt{2}} \text{ drop.}} \text{ dB}$$

☆ At $\omega \gg \omega_c$, $\phi \approx -90°$, and

$M = \frac{K}{\sqrt{\tau^2\omega^2+1}}$. So, $M = \frac{K}{\sqrt{\left(\frac{\omega}{\omega_c}\right)^2 + 1}}$

$$M_{dB} = 20\log_{10}K - 20\log_{10}\sqrt{\left(\frac{\omega}{\omega_c}\right)^2 + 1}$$

$$\approx 20\log_{10}K - 20\underbrace{\log_{10}\left(\frac{\omega}{\omega_c}\right)}_{\text{number of decades.}}$$

So slope is -20 dB/decade.

A 1^{st} order system $G(s) = \frac{1}{\tau s+1}$ is a "low-pass filter".

The corner frequency is $\omega_c = \frac{1}{\tau}$. So $G(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)+1}$

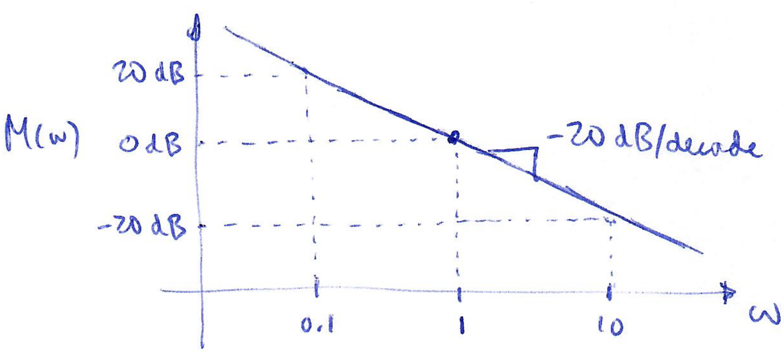
and $G(j\omega) = \frac{1}{j\left(\frac{\omega}{\omega_c}\right)+1}$. If $\omega \ll \omega_c$, $G(j\omega) \approx 1$, so unity gain and no phase shift. These low frequencies can "pass" through. Hence the name "low-pass".

Frequencies above the corner, $\omega \gg \omega_c$ are attenuated. For every decade above ω_c, we get roughly another -20 dB attenuation (multiply by $\frac{1}{10}$).

Ex integrator. Consider $G(s) = \frac{1}{s}$, a pure integrator

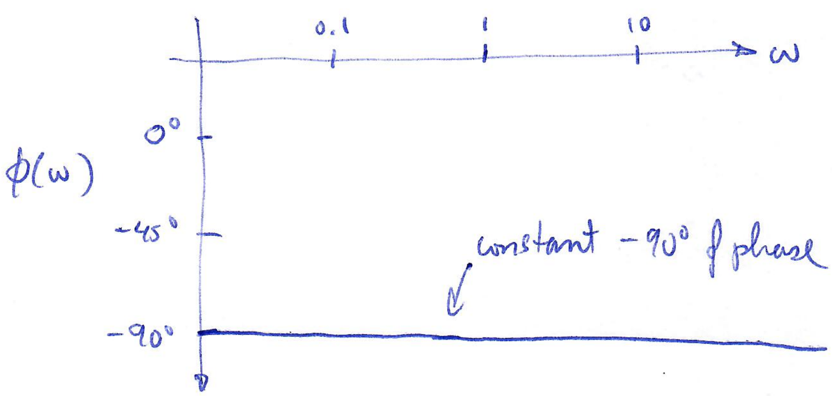
$G(j\omega) = \frac{1}{j\omega} = -j\left(\frac{1}{\omega}\right)$. So $M(\omega) = \frac{1}{\omega}$ and $M_{dB} = -20\log_{10}\omega$

and $\phi(\omega) = -90°$ (since $\frac{-j}{\omega}$ is pure neg. imaginary)



Integrators amplify low frequencies (think about integrating a constant... steady-state goes to ∞!)

but they attenuate high frequencies.



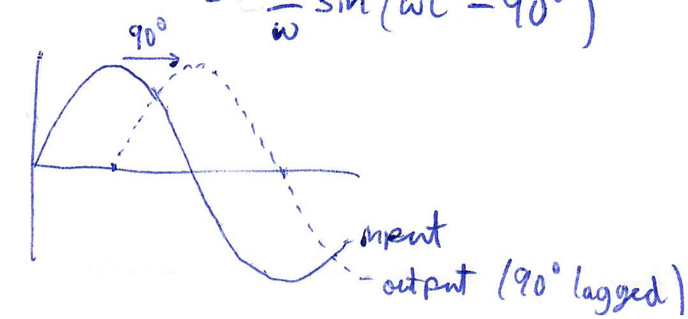
output occurs 90° after the input, so it is lagging by 90°. (phase = $-90°$)

We can test the phase easily:

if $u(t) = \sin\omega t$, then

$y(t) = \int \sin(\omega t)\,dt = -\frac{1}{\omega}\cos\omega t$

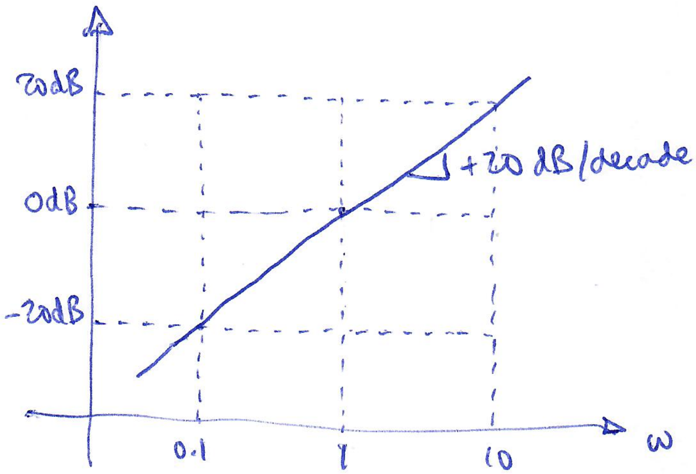
$= \frac{1}{\omega}\sin(\omega t - 90°)$



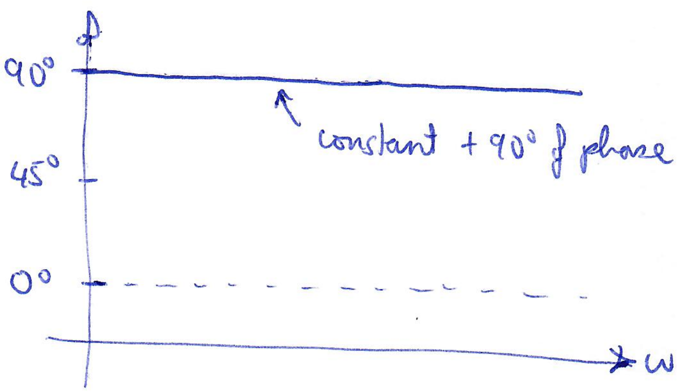
Ex differentiator. Consider $G(s) = s$, a pure derivative.

$G(j\omega) = j\omega$. So $M(\omega) = \omega$ and $M_{dB} = 20 \log_{10} \omega$

and $\phi(\omega) = 90°$ (since $\omega > 0$ and $j\omega$ is pure imaginary).

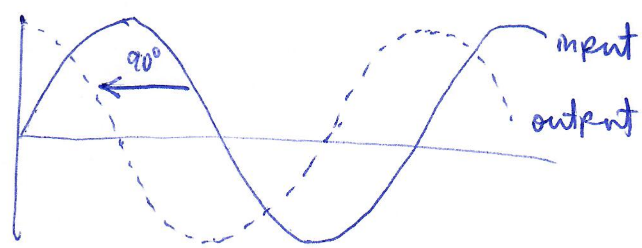


differentiators amplify high frequencies (derivative of noise blows up!)
but they attenuate low frequencies (think about differentiating a constant.)



We can verify this as well:
if $u(t) = \sin \omega t$, then

$$y(t) = \frac{d}{dt} u(t) = \omega \cos \omega t$$
$$= \omega \sin(\omega t + 90°)$$



output occurs 90° <u>before</u> the input, so it is <u>leading</u> by 90°. (phase = +90°)

Bode plots can be quickly produced in Matlab using the bode(...) command. ex:

```
s = tf('s');
bode(s)
```

produces the plot above.